

Haldane Topological Orders in Motzkin Spin Chains

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Motzkin spin chains are frustration-free models whose ground-state is a combination of Motzkin paths. The weight of such path contributions can be controlled by a deformation parameter t . As a function of the latter these models, beside the formation of domain wall structures, exhibit a Berezinskii-Kosterlitz-Thouless phase transition for $t=1$ and gapped Haldane topological orders with constant decay of the string order parameters for $t < 1$. By means of numerical calculations we show that the topological properties of the Haldane phases depend on the spin value. This allows to classify different kinds of hidden antiferromagnetic Haldane gapped regimes associated to nontrivial features like symmetry-protected topological order. Our results from one side allow to clarify the physical properties of Motzkin frustration-free chains and from the other suggest them as a new interesting and paradigmatic class of local spin Hamiltonians.

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Spin chains play a crucial role in many fundamental physical phenomena like magnetism [1], quantum phase transitions [2], topological orders [3] and quantum computation [4]. A fundamental contribute to the understanding of spin chains is provided by the seminal papers by Haldane [5] where a new topological phase, the Haldane phase (HP), uniquely detectable via a non-local string order parameter [6] has been discovered for spin-1 XXZ Heisenberg chains. This has driven significant efforts to look for new kinds of models whose topological order can be described in terms of a string order parameter [7] motivating the discovery of the celebrated Affleck-Kennedy-Lieb-Tasaki (AKLT) model [8]. Although the argument of Haldane is given for integer spin chains, only integer spin XXZ-like and AKLT-like chains own topological HP and is therefore non-trivial to find and study new classes of Hamiltonians where HP emerges.

Thanks to the strongest quantum "resource", namely the entanglement, spin models have also a fundamental role in the simulation of quantum logical gates for quantum computation [4]. For this reason finding and studying Hamiltonians with highly entangled spins is currently one of the most challenging and intriguing fields in quantum physics.

In this direction local integer frustration-free spin Hamiltonians whose ground-state can be expressed as a combination of Motzkin paths [9] have been recently introduced [10, 11]. Among others interesting aspects, their importance is given by the fact that they own a level of entanglement entropy which strongly exceeds the one exhibited by other previously known local models. Relevantly, also for half-integer spins, a similar class of Hamiltonians, the Fredkin spin chains, exhibiting the same features [12, 13] has been introduced. In addition to their entanglement properties Motzkin chains own further very peculiar properties. Indeed, even if they are purely local models, for high spin values s (i.e., $s \geq 2$) they behave as *de facto* long range Hamiltonians being able to violate cluster decomposition properties (CDP) and area law (AL) scaling of the entanglement entropy [12]. Very recently, a deformed

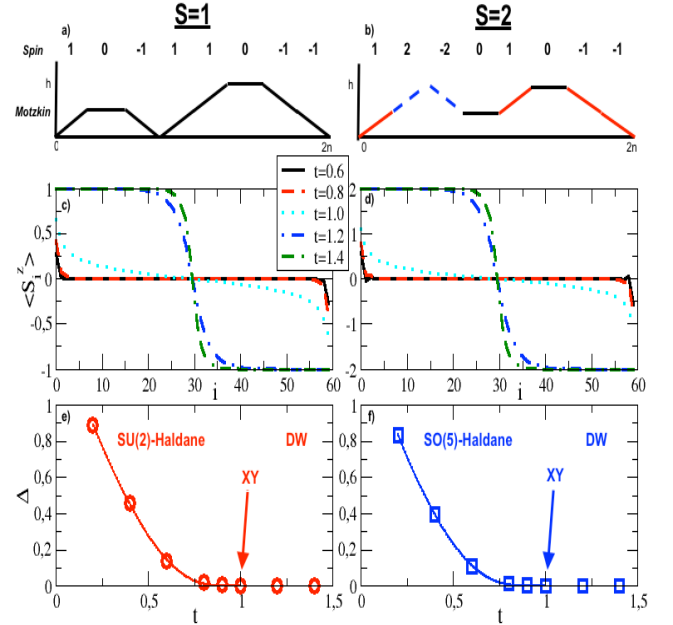


Figure 1: (Color online) *Upper panels*: Cartoons of a possible Motzkin path and its representation in terms of spins for the two cases a) uncolored $s = 1$ and b) colored $s = 2$. *Central panels*: DMRG local magnetization for a system of length $2n = 60$ at different t deformation values $\langle S_i^z \rangle$ for c) $s = 1$ and d) $s = 2$. *Lower panels*: Thermodynamic limit of the gap $\Delta = E_1 - E_0$ as a function of t for e) $s = 1$ (red circles) and f) $s = 2$ (blue squares). The continuous lines are fitted with the form $\sim (\exp -b/\sqrt{|t - t_c|})$ with $t_c = 1$ and b a fitting parameter. The thermodynamic limit is extrapolated by using chains of length up to $2n = 60$. All the DMRG simulation are performed by keeping at most 1024 DMRG states and 5 finite size sweeps with an error energy $< 10^{-9}$ (10^{-7}) for $s = 1$ ($s = 2$).

version of Motzkin [14] and of Fredkin [15] chains have been introduced, and their gap studied [16], with the contribution of Motzkin or Fredkin paths to the ground-state being weighted through the introduction of a parameter t .

Due to the aforementioned arguments it appears clear as these new models are both very interesting by themselves and they could open the path towards fundamental applications. This motivates us to investigate a Motzkin chain for different spin values and in presence of path deformations. Here after an introduction of the model in terms of deformed Motzkin paths, we present density matrix renormalization group (DMRG) [17] calculations which allow to reveal the appearance of different phases as a function of the deformation parameter t . In particular we show that local magnetization is able to capture the $t > 1$ regime where a clear domain wall structure takes place independently by the spin value s . From the other side once $t < 1$ the system undergoes to a phase transition of a Berezinskii-Kosterlitz-Thouless (BKT) type [18] as signaled by an exponential opening of the gap. Moreover our numerical calculations confirm that for this kind of deformation the entanglement entropy is bounded and size independent [14]. Crucially we find that this gapped regime can be described solely by a non-vanishing value of string order parameters thus showing the topological nature of the $t < 1$ deformed Motzkin chains. For $s = 1$ only one string is finite, similarly to what happens in the $SU(2)$ -Haldane phase for XXZ or AKLT spin-1 models, thus revealing the presence of a symmetry-protected topological (SPT) order. On the other hand, for $s = 2$ different kinds of Haldane phases have been obtained [19, 20]. In particular, for the spin 2 Motzkin chain, we show that two strings display a constant decay exhibiting a phase similar to the $SO(5)$ -topological Haldane order occurring in $s = 2$ AKLT model [21]. Interestingly, unlike the undeformed case $t = 1$, for $t < 1$ the CDP [22] is valid.

Model. The spin model we consider has the peculiarity of having a ground state which can be expressed in terms of Motzkin paths describing all the possible $2n$ moves that one can make to go from a point of height $h = 0$ to another point of the same h without crossing the 0 line [10, 11]. As shown in Fig. 1 a) and b), spins can be seen as moves by imposing that up/zero/down spin corresponds to increasing/conserving/decreasing the height of the path. Of course, for spin $s = 1$ only uncolored steps (uncolored Motzkin chain) are allowed and the system can be seen as a spin $s = 1$ chain, while larger values of s can be achieved when colored steps are possible (colored Motzkin chain). The Hamiltonian reads:

$$H = \sum_{j=1}^{2n-1} \Pi_{j,j+1}(s, t) + \Pi_{\partial}(s) + \sum_{j=1}^{2n-1} \Pi_{j,j+1}^{cross}(s) \quad (1)$$

where $\Pi_{j,j+1}(s, t) = \sum_{k=1}^s (|\phi(t)^k\rangle\langle\phi(t)^k|_{j,j+1} + |\psi(t)^k\rangle\langle\psi(t)^k|_{j,j+1}) + |\Theta(t)^k\rangle\langle\Theta(t)^k|_{j,j+1}$ is the bulk term, $\Pi_{\partial}(s) = \sum_{k=1}^s |-k\rangle\langle-k|_1 + |k\rangle\langle k|_{2n}$ is the boundary term which makes more favorable for the first spin to point upward, $|k\rangle$, and the last downward, $|-k\rangle$. The latter term in Eq. (1) $\Pi_{j,j+1}^{cross}(s) = \sum_{k \neq k'} = |k, -k'\rangle\langle k, -k'|$, present only for $s > 1$, ensures the color matching of up and down spins with the same height. The parameter t appearing in $\Pi_{j,j+1}(s, t)$ describes path de-

formations and $|\phi(t)^k\rangle = (1 + t^2)^{-1/2}(|k, 0\rangle - t|0, k\rangle)$, $|\psi(t)^k\rangle = (1 + t^2)^{-1/2}(|0, -k\rangle - t|-k, 0\rangle)$ and $|\Theta(t)^k\rangle = (1 + t^2)^{-1/2}(|k, -k\rangle - t|0, 0\rangle)$. The deformation induced by $t \neq 1$ keeps the model frustration-free [14], and, while for $t = 1$ we recover the undeformed model [10–12], for $t > 1$ ($t < 1$) the paths having larger (smaller) h are favored in the ground state.

$t \geq 1$ Regime. This latter point explains the $t > 1$ behavior of the local magnetization $\langle S_j^z \rangle$ observed in Fig. 1 c) and d) for $s = 1$ and $s = 2$ respectively. Indeed, since $t > 1$ makes more probable higher paths, in terms of spins this corresponds to a domain wall (DW) structure where the up and down spins are separated in two different regions of equal length n [23] and the zero spins are basically absent. Relevantly, as shown in Fig. 1 e) and f), this latter regime is gapless ($\Delta = 0$) meaning that the difference between the ground-state E_0 and the first excited state E_1 energy goes to zero in thermodynamic limit (TDL). The aforementioned features allow to find the analogy between Eq. 1 and the $\gamma < 1$ regimes of XXZ chains for both spin 1 and 2 (being γ the z -anisotropy parameter [24]). Further similarities can be also noticed for the $t = 1$ case where a gapless regime is associated to a power-law decay of the correlation function $\langle S_i^+ S_j^- \rangle$ and, as exactly shown in [12], an exponential decay of $\langle S_i^z S_j^z \rangle - \langle S_i^z \rangle \langle S_j^z \rangle$ [25] thus resembling the XY phase of XXZ models but, with the key feature that both AL decay and CDP are violated for $s = 2$.

$t < 1$ Regime. From the other side, as already mentioned, a $t < 1$ deformation minimizes the height of the possible paths. This is clearly visible in the $\langle S_j^z \rangle$ behavior shown in Fig. 1 c) and d) where an almost totally flat local magnetization with $h = 0$ is observed. Crucially $\langle S_j^z \rangle$ shows also antiparallel peaks at the edges of the chain thus supporting the possible presence of edge states. This effect, as explained before, is produced by the $\Pi_{\partial}(s)$ term in Eq. (1) which has also the role of breaking the ground-state degeneracy. The almost flat magnetization can explain another behavior, namely that the entanglement entropy, $S(A) = -\text{Tr} \rho_A \log_2 \rho_A$ of a subsystem A , is bounded and does not depend on neither the chain nor on the partition length [14], meaning that the area law scaling is fulfilled. Indeed, as it is possible to see in Fig. 2 a), we find that $S(A)$ is constant at fixed t for any $2n$ while it grows almost linearly with the deformation strength. The latter is easily explained by the fact that for $t < 1$ the strength of t actually affects only the first and the last move with flat $\langle S_j^z \rangle = 0$ in the bulk. Consequently a larger/smaller t will produce an higher/lower value of $|\langle S_i^z \rangle|$ in the first and last site thus generating more/less entropy. Notice that, as shown in Fig. 2 a), this behavior holds for any considered s value. A less trivial aspect, conjectured in [14], emerges by looking Fig. 1 e) and f), namely $t < 1$ deformations support the presence of finite gap in the thermodynamic limit. As visible in the same figures, for both $s = 1$ and $s = 2$ the gap opens compatibly with an exponential decay $\Delta \sim \exp -b/\sqrt{|t - t_c|}$, being $t_c = 1$ and b a fitting parameter, thus signaling a BKT-like phase transition. In integer spin chains gapped regime can be usually associated to either antiferromagnetic (AF) order de-

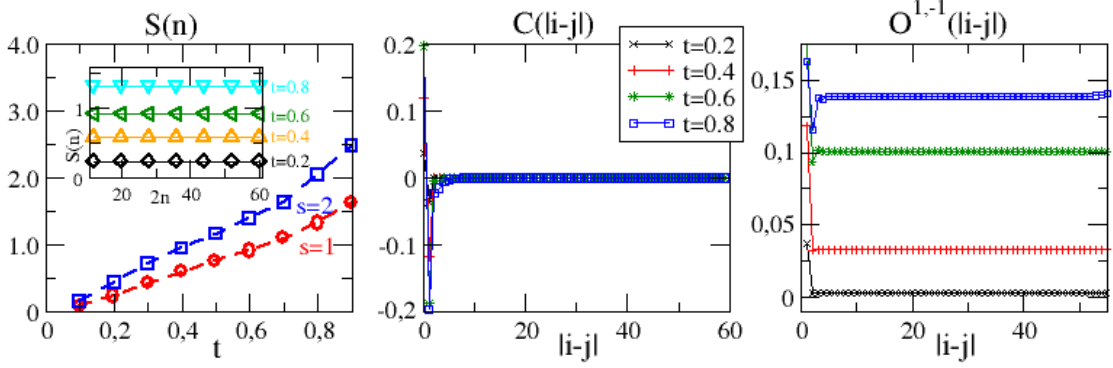


Figure 2: (Color online) a): Entanglement entropy $S(A)$ for a subsystem having length n with $1 \leq i \leq n$ for $s = 1$ (red symbols) and $s = 2$ (blue symbols). The inset shows the constant behavior of $S(A)$ as a function of the size $2n$. b): $C(|i-j|)$ for different $t < 1$ values. c): $O^{1,-1}(|i-j|)$ for different $t < 1$ values. The correlations in the panels b) and c) are evaluated in a system of size $2n = 60$ with i pinned in the first chain site. We checked that different i -values do not alter the physical behavior of the correlations.

scribed by the two points correlation functions

$$C(|i-j|) = \langle S_i^z S_j^z \rangle - \langle S_i^z \rangle \langle S_j^z \rangle \quad (2)$$

or to Haldane orders described by a string order parameter

$$O^{k,\bar{k}}(|i-j|) = \langle L_i^{k,\bar{k}} e^{i\pi \sum_{i \leq \ell < j} L_\ell^{k,\bar{k}}} L_j^{k,\bar{k}} \rangle \quad (3)$$

where $L^{k,\bar{k}} = |k\rangle\langle k| - |-k\rangle\langle -k|$. Notice that, for $s = 1$, $k(\bar{k})$ can be solely equal to $1(-1)$ thus $L_i^{1,-1} = S_i^z$ while for $s = 2$, $k(\bar{k})$ can take the values $1(-1)$ and $2(-2)$ and $S_i^z = 2L_i^{2,-2} + L_i^{1,-1}$. The important informations encoded in such non-local order parameters is that, once it is finite, Eq. (3) describes a topological phase, usually called HP, with hidden antiferromagnetism. The hidden AF order is given by the fact that it can not be described by usual two point correlation functions Eq. (2) thus describing a phase where spins up and down are rigorously alternated and separated by a random number of zero spins. Of course, while for $s = 1$ the HP can be given only by alternating $+1$ and -1 spins thus signaled by a finite $O^{1,-1}(|i-j|)$, for $s = 2$ the hidden order can be signaled, as it will be clear in the following, by two or even solely one finite $O^{k,\bar{k}}(|i-j|)$.

$s = 1$ case. Here we start our analysis with the $s = 1$ case by evaluating both $O^{1,-1}(|i-j|)$ and $C(|i-j|)$ for different $t < 1$ values. Fig. 2 clearly shows that while $C(|i-j|)$ rapidly decays to zero the string order parameter remains constant as a function of the distance thus signaling the presence of a HP. This aspect, in analogy with XXZ chains, supports our prediction regarding the BKT nature of the phase transition. We also checked that the strings along transverse directions decay. Furthermore, as visible in Fig. 2 c), the string $O^{1,-1}(|i-j|)$ saturates to a constant value which becomes bigger the larger is t . At a first look this aspect could seem counterintuitive since one expects that the larger is the gap, the stronger is the string order. Nevertheless an easy interpretation of the $O^{1,-1}(|i-j|)$ behavior as function of t comes

by the geometrical meaning of deformations. Indeed, as explained before, a small t favors the paths with low h . Intuitively, one can argue that the path with smaller h is the one where the first and last move corresponds to respectively the rising and the lowering steps with in the middle a series of flat moves. This means that the number of $+1$, -1 spins producing the hidden AF order is minimized by reducing t , thus producing a lower saturation value of the string order. Nevertheless we checked that even very small deformations support the presence of a constant $O^{1,-1}(|i-j|)$, suddenly disappearing for $t = 1$, thus allowing to unambiguously conclude that the uncolored $t < 1$ Motzkin chain has topological order with hidden AF. This phase is usually called $SU(2)$ -Haldane phase and it has been observed both in spin-1 XXZ [6, 26] chains and in the AKLT model [27]. We will keep this nomenclature even if for our model only the operator $\sum_i S_i^z = \sum_i L_i^{1,-1}$ commutes with the Hamiltonian. This is similar to what happens in the spin-1 XXZ model when a single ion anisotropy term is included, breaking the $SU(2)$ invariance but preserving the Haldane phase.

For systems with conformal invariance like the spin-1 XXZ model, the topological order is also captured by an even degeneracy of the entanglement spectrum (ES) [28]. From the other side, when Hamiltonians cannot be described by conformal field theory (CFT), like for instance the AKLT [29] or exotic bosonic Hamiltonians [30], the ES does not present even degeneracy but the topological order is assured by the presence of edge modes and finite strings. We checked that due to the lack of a possible description in terms of CFT [11] the ES of $t < 1$ deformed Motzkin chains does not display any degeneracy. Nevertheless edge modes, visible in Fig. 1 c) and d), and the finite string in Fig 2 assure the topological order. The latter has a further fundamental property due to fact that it appears for an odd value s of the spin. Indeed, once a Haldane phase takes place for odd spins s , SPT order [7] is generated. This is given by the fact that the edge modes fractionalize in two half-integer spins which cannot be removed

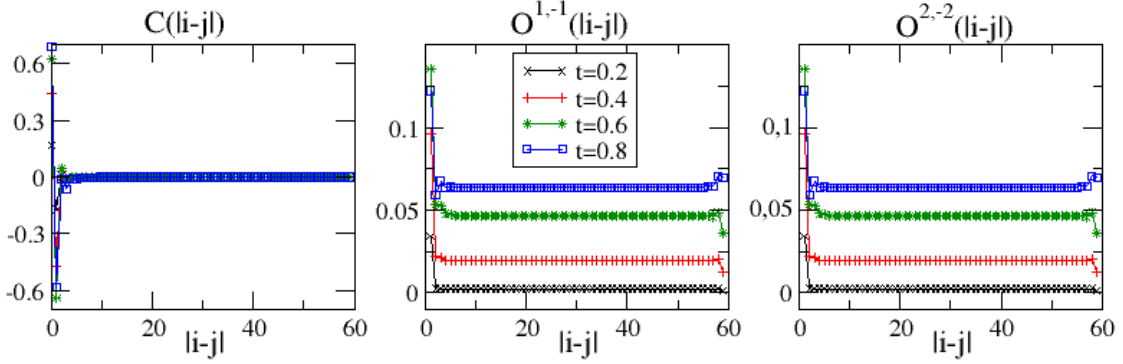


Figure 3: (Color online) a): $C(|i-j|)$ for different $t < 1$ values. b): $O^{1,-1}(|i-j|)$ for different $t < 1$ values. c): $O^{2,-2}(|i-j|)$ for different $t < 1$ values. All the correlation are evaluated in a system of size $2n = 60$ with i pinned in the first chain site. We checked that using different values of i do not alter the physical behavior of the correlations.

unless in presence of a phase transition or an explicitly symmetries breaking. This consideration allows to conclude that the $s = 1$ version of Eq. (1) with $t < 1$ deformations supports the presence of SPT topological order with bounded and size independent entanglement thus strongly characterizing the Motzkin chains.

$s = 2$ case. As shown in [11, 12], the $s > 1$ undeformed $t = 1$ chains have much more intriguing properties with respect to lower spin case. These are induced by the presence of colors which increase the symmetry of the system. As for the $s = 1$, $s = 2$ XXZ Heisenberg models and AKLT models can support the presence of gapped phases for positive z -anisotropies. The gap can again be associated to AF order detected by $C(|i-j|)$ or to different kinds of Haldane order, see for instance [19, 31] and reference therein. In particular in such systems SPT topological order is signaled by finite values of both $O^{1,-1}(|i-j|)$ and $O^{2,-2}(|i-j|)$. Moreover, as conjectured in [32] and shown in [31, 33], single ion anisotropy terms can support the formation of a SPT $SU(2)$ -Haldane order even for $s = 2$. Our calculations in Fig. 1 f) show that again the colored Motzkin chain is gapped for $t < 1$ and the gap is associated to a hidden AF order since $C(|i-j|)$ has a clear exponential decay rapidly saturating to 0 as shown in Fig. 3 a). From the other side both $O^{1,-1}(|i-j|)$ and $O^{2,-2}(|i-j|)$ have a constant and basically equal behavior thus clarifying that the $s = 2$ Motzkin chains with $t < 1$ deformations support the presence of an $SO(5)$ -Haldane phase. It is worth stressing that $SO(5)$ is not the symmetry of our model, rather $U(1) \times U(1) \times \mathbb{Z}_2$, since only $\sum_i L_i^{2,-2}$ and $\sum_i L_i^{1,-1}$ commute with the Hamiltonian, like in the $s=2$ AKLT model when the term $\sum_i (S_i^z)^2$ is switched on. Also in that case $SO(5)$ -Haldane phase survives once the symmetry is lowered from $SO(5)$ to $U(1) \times U(1)$ [34]. In our case the symmetry is supplemented by the invariance under interchanging the two colors (\mathbb{Z}_2). This is the reason why $O^{1,-1}(|i-j|)$ and $O^{2,-2}(|i-j|)$ are the same, as shown in Fig. 3. Moreover it is important to notice that, in analogy

with the $s = 1$ case, the strings become stronger by increasing t . Fig. 3 b) also shows a further information encoded in the $C(|i-j|)$ behavior. Indeed, on the contrary to the $t = 1$ regime [12], its exponential decay is associated to a 0 edge-to-edge value thus holding the CDP. The opening of a gap in a colored Motzkin chain, therefore, restores the pure locality of the model in Eq. (1), in agreement with the general findings for gapped local Hamiltonians [35].

Conclusions and Perspectives. In conclusion, our results demonstrate the existence of topological Haldane orders in a new class of spin Hamiltonians. Furthermore we shown the behavior of Motzkin chains as a function of the deformation strength t . While the undeformed $t = 1$ case has XY -like features, for $t > 1$ the system presents a gapless domain wall structure. From the other side at $t = 1$ a BKT-like phase transition, characterized by a exponential opening of the gap, occurs for any $t < 1$ values. The gapped regime is associated to SPT hidden Haldane antiferromagnetic orders signaled by finite values of string non-local order parameters. The two possible Haldane orders have the peculiarity of having an entanglement entropy independent from both block and chain size. Moreover our results suggest that it would be very interesting to have a physical implementation of the Motzkin spin chains. In this respect cold atomic systems, which have been already proposed to simulate several kinds of spin Hamiltonians with topological orders [36], could provide a possible physical platform to implement Motzkin chains. Their experimental realization could be relevant for technological achievements since, from one side, symmetry-protected topological orders have been proposed as ideal candidates towards the realization quantum devices like quantum repeaters [37] and substrate for measurement-based quantum computation [38] while, from the other, side Motzkin paths may have applications in the field of polymer absorption [39]. Finally we underline that, in future works, it would be very interesting to study the gapped regimes in the fermionic version of the $s = 3/2$ Fredkin model where exotic Haldane regimes can take place [40].

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